Two Reconstruction Algorithms of the Process on the Output of the Exponential Converter

Vladimir Kazakov[1], Mónica Sedeño[2]

[1] Department of Telecommunications, ESIME Zac-IPN, Mexico D.F., Mexico
{Vladimir Kazakov}, vkazakov41@hotmail.com,
{Mónica Sedeño}, msedeno@ipn.mx

Abstract. The two extrapolation algorithms of the Sampling – Reconstruction Procedure of the non Gaussian random process at the output of the exponential converter are considered. The input process is the call the Gaussian process. The conditional mean rule is used for the investigation. The principal SRP characteristics (the reconstruction function and the error reconstruction function) are obtained. The comparison of the optimal and non optimal algorithms is given in this investigation.

Keywords: Conditional Mean, Conditional Variance, Covariance Function, Extrapolation.

1 Introduction

The classical Sampling Theorem associated with the names of Whittaker, Kotelnikov and Shannon has been proved for deterministic functions.

The Balakrishnan's theorem [1] has been proved for stationary random processes with limited spectrum. This theorem has some important disadvantages [2]:

i) The principal characteristic - probability density function (pdf) - is not taken into account.
ii) The Reconstruction Function is considered as a linear function with the samples for any kind of stochastic process.
iii) The Basic Function sin x /x is the same for any kind of stochastic process.
iv) The error reconstruction function is cero for any kind of stochastic processes.
v) The number of samples is equal to infinite.
vi) It is impossible to describe the optimal extrapolation procedure for arbitrary stochastic processes.

In fact, it is very important to consider the Sampling- Reconstruction Procedure (SRP) for stochastic processes with a finite number of samples. We want to obtain the two principal characteristics of SRP: 1) the reconstruction function and 2) the error

© L. Sánchez, O. Pogrebnyak and E. Rubio (Eds.)
Industrial Informatics
Research in Computing Science 31, 2007, pp. 253-260

reconstruction function for some arbitrary stochastic processes. This problem can be resolved by the application of the famous statistical rule of random variables - the Conditional Mean Rule [2].

If we know an arbitrary number of samples, it is possible to say that the Conditional Mean Function is the Reconstruction Function and the Conditional Variance is the Error Reconstruction Function. On the basis of this approach the SRP of different types of stochastic processes have been investigated [3 - 8].

The present paper is devoted to the investigation of the SRP of the non Gausssian process at the output of the exponential converter, driven by the Gaussian process. We analyze optimal and non optimal SRP algorithms of the mentioned process and make the comparison of these two algorithms.

The optimal SRP algorithm is based on the knowledge of the exact expression of pdf of the process under consideration. The non optimal SRP algorithm is characterized by the covariance function of this process. In both cases we use the extrapolation case of the reconstruction.

2 Gaussian Case

Let us consider the general case of the stationary Gaussian process $x(t)$ with the mathematical expectation m, the variance σ^2, and the covariance function $K(t_i-t_j)$. On the basis of the covariance function one can write the covariance matrix:

$$\left|K(t_i-t_j)\right| = \begin{vmatrix} K(t_1-t_1) & K(t_1-t_2) & \cdots & K(t_1-t_m) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_m-t_1) & K(t_m-t_2) & \cdots & K(t_m-t_m) \end{vmatrix}. \quad (1)$$

These characteristics are complete because they determine the well known analytical expression of the multidimensional Gaussian pdf. There is the expression for the conditional Gaussian multidimensional pdf. On the basis of this conditional pdf one can determine the expressions for the conditional mathematical expectation $\tilde{m}(t)$ and for the conditional variance $\tilde{\sigma}^2(t)$. We fix an arbitrary set of N samples $X,T=\{x(T_1),x(T_2),...,x(T_N)\}$ where $x(T_i)(i=1,...,N)$ the value of the sample is at the time moment T_i. The expressions for $\tilde{m}(t)$ and $\tilde{\sigma}^2(t)$ have the view:

$$\tilde{m}(t) = m + \sum_{i=1}^{N}\sum_{j=1}^{N} K(t-T_i)a_{ij}\left[x(T_j)-m(T_j)\right]. \quad (2)$$

$$\tilde{\sigma}^2(t) = \sigma^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} K(t-T_i)a_{ij}K(T_j - t). \tag{3}$$

where a_{ij} is the element of the inverse covariance matrix (1) with the change of the designation - T_i, T_j instead t_i, t_j:

$$|a_{ij}| = |K(t_i - t_j)|^{-1}, \quad \sum_{j=1}^{m} a_{ij}K(t_j - t_i) = 1. \tag{4}$$

Let us consider the simple case $m=0$ and $\sigma^2=1$, then instead of (2) we have:

$$\tilde{m}(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} K(t-T_i)a_{ij}x(T_j) = \sum_{j=1}^{N} x(T_j)b_j(t) \ . \tag{5}$$

where the basic function $b_j(t)$ is determined by the following expression:

$$b_j(t) = \sum_{i=1}^{N} K(t-T_i)a_{ij} \ . \tag{6}$$

From (5) and (6) one can see that the reconstruction function $\tilde{m}(t)$ is the linear function of samples and the basic function is not the function $Sinx/x$. The optimal basic function $b_j(t)$ depends on the number of the sample j, it depends on the quantity of samples N, and it depends on the different covariances between the sections of the process in T_i and T_j moments (the elements of the inverse covariance function a_{ij} includes the corresponding covariance moments $K(T_i-T_j)$, the covariance $K\ (t-T_i))$ between the current section in the time t and the section in the time of the samples T_i.

The formula (3) shows that the error reconstruction function does not depend on the values of samples, but it depends on the time moments of sample T_i.

One example of the easiest case is the Markov process, which is generated at the output of an integrator circuit RC with the parameter $\alpha=1/RC$. And its own covariance function is expressed with the exponential form:

$$K(\tau) = \sigma^2 \exp(-\alpha|\tau|). \tag{7}$$

The covariance time $\tau_c = 1/\alpha$ will be chosen in this case as $\tau_c = 1$. The interpolation process depends on the samples which are closer $x(T_i)$ and $x(T_{i-1})$ $(T_i < t < T_{i-1})$ and the extrapolation process is described by the latest sample $x(T_N)$, $(t > T_N)$. These characteristics are determined by the Markov properties.

3 The output process of the exponential converter

Let us consider a non linear exponential converter with the characteristic:

$$\eta(t) = a_0\, e^{\beta \xi(t)} \; . \tag{8}$$

where a_0 and β are constants, $\xi(t)$ is an input process, $\eta(t)$ is an output process.

We suppose that the process $\xi(t)$ is Gaussian Markov process with the covariance function (7) and we put $\sigma^2 = 1$.

Firstly, we determine the unconditional characteristics of the process $\eta(t)$. Using the known methodology:

$$F(\eta) = f(h(\eta)) \frac{1}{\left|\frac{d\eta}{d\xi}\right|} \; . \tag{9}$$

where $h(\eta)$ is an inverse function with respect to (8).

Then, we can obtain the inverse unconditional characteristics of the process $\xi(t)$

$$\xi(t) = \frac{1}{\beta} \ln\left(\frac{\eta(t)}{a_0}\right). \tag{10}$$

After that, we derivate the expression (8) with respect of $\xi(t)$

$$\left|\frac{d\eta}{d\xi}\right| = a_0 \beta \exp\left(\beta \xi(t)\right). \tag{11}$$

We get an one-dimensional pdf $F(\eta)$.

$$F(\eta) = \frac{1}{a_0 \beta \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{1}{\beta} \ln \frac{\eta}{a_0}\right)^2 - \ln \frac{\eta}{a_0} \right]. \tag{12}$$

The graph of this function is presented in Fig. 1.

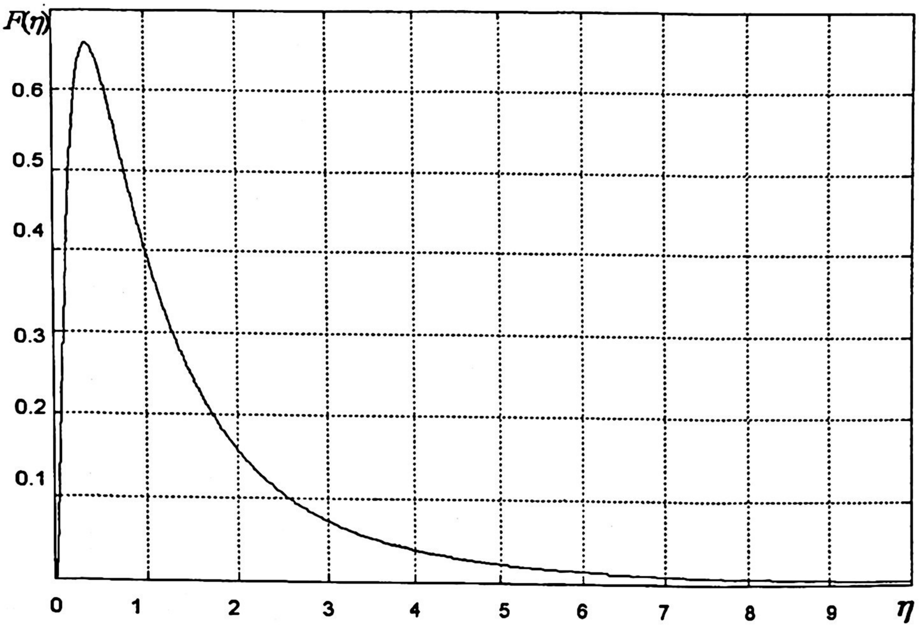

Fig. 1. Pdf of the output process with the parameters $a_0=1,\ \beta=1$

On the basis (10) we can determine the mathematical expectation m_η

$$m_\eta = m_1^{\eta} = a_0 \exp\left(\frac{\beta^2}{2}\right). \tag{13}$$

and the variance

$$\sigma_\eta^2 = a_0^{\,2}\left[\left(\exp 2\beta^2\right)-\left(\exp \beta^2\right)\right]. \tag{14}$$

Using (11) and (12) for $\beta = a_0 = 1$ we obtain the concrete values of the mean and the variance: m_η=1.6487, $\sigma_\eta{}^2$= 4.6708.

Writing (8) for two time moments t and $t + \tau$, multiplying both expressions and applying the average operation, one can find the covariance function of de output process $\eta(t)$:

$$K_{\eta}(\tau) = a_0^2 e^{2\alpha a_0} \exp[\alpha^2 \sigma_{\xi}^2 (1 + K_{\xi}(\tau))] \ . \tag{15}$$

The algorithm of extrapolation only depends on the sample $\eta(T_N)$ because $\eta(t)$ is a Markov process.

4 The optimal reconstruction

Let us consider the optimal extrapolation procedure. Using the method [6] on the basis (10) – (13) we obtain the required expressions:

$$\tilde{m}_1^{\eta}(t) = a_0 \exp\left\{\beta \tilde{m}_1^{\xi}(t) + \frac{1}{2}\beta^2\right\} . \tag{16}$$

$$\tilde{m}_2^{\eta}(t) = a_0^2 \exp\left\{2\beta \tilde{m}_1^{\xi}(t) + \beta^2 + \beta^2 \tilde{\sigma}_{\xi}^2(t)\right\} . \tag{17}$$

Here the first conditional moment function $\tilde{m}_1^{\xi}(t)$ is determined by the formula (5) with $N=1$:

$$\tilde{m}_1^{\xi}(t) = \xi(T_N) \exp(-\alpha(t - T_N)) . \tag{18}$$

The value of the input sample $\xi(T_N)$ is connected with the value of the output sample $\eta(T_N)$ by the inverse expression

$$\xi(T_N) = \frac{1}{\beta} \ln \frac{\eta(T_N)}{a_0} . \tag{19}$$

We obtain the conditional variance $\tilde{\sigma}_{\eta}^2(t)$ by the formula

$$\tilde{\sigma}_{\eta}^2(t) = \tilde{m}_2^{\eta}(t) - [m_1^{\eta}(t)]^2 . \tag{20}$$

where the right part is determined by the expressions (14) – (17).
We can see that the conditional variance $\tilde{\sigma}_{\eta}^2(t)$ depends on the values of samples in the contrast with the Gaussian case.
The results of the calculations of the error reconstruction function are presented in Fig. 2. Here there is the family of the curves $\tilde{\sigma}_{\eta}^2(t)$ for some different values of

output samples $\eta(T_N)$ = 0.25, 1, 2, 3, 4, 5. As one can see any chosen sample has its own special curve. All the curves converge at the value of the variance, which is $\sigma_\eta^2 = 4.67$. These curves characterize the minimum error reconstruction function for the optimal reconstruction algorithm of the extrapolation type.

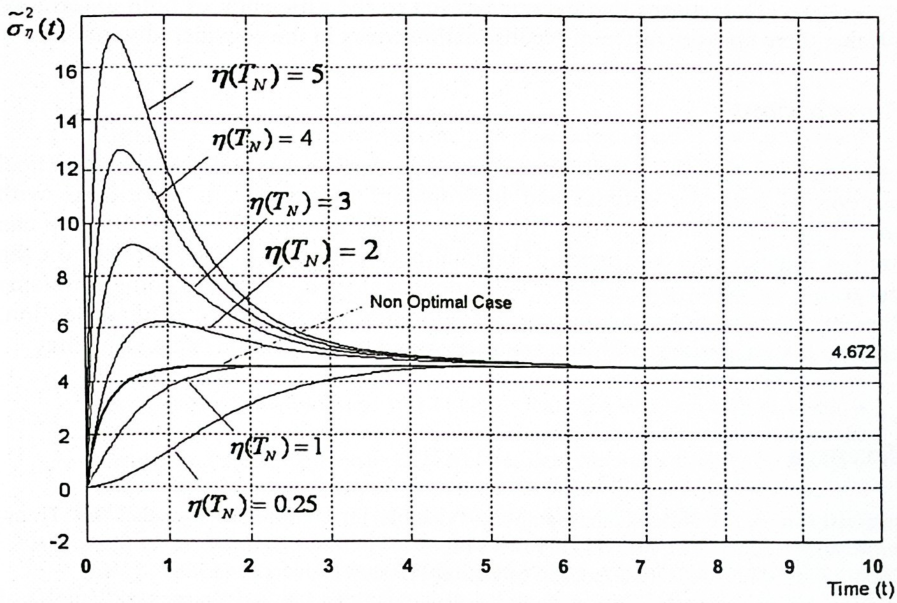

Fig. 2. The graphs of the Reconstruction Error Function. Comparison between the Optimal Case and the Non-optimal Case.

5 The non Optimal Case

Let us consider another extrapolation algorithm of the reconstruction. This algorithm is based on the knowledge of the covariance output function (13) only. It means that we must use the Gaussian approximation for the SRP description automatically, i.e. it is necessary to apply the formulas (2) and (3) with $N = 1$. Then we have:

$$\tilde{m}_\eta(t) = m_\eta + K_\eta\left(t - T_N\right)\sigma_\eta^{-2}\left[\eta(T_N) - m_\eta\right]. \tag{21}$$

$$\tilde{\sigma}^2_\eta(t) = \sigma_\eta^2\left[1 - \sigma_\eta^{-2} K_\eta^2\left(t - T_N\right)\right]. \tag{22}$$

Substituting (11) – (13) we obtain the reconstruction function $\tilde{m}_\eta(t)$ and the error reconstruction function $\tilde{\sigma}_\eta^{\,2}(t)$. The graph of the function $\tilde{\sigma}_\eta^{\,2}(t)$ is shown in Fig. 2 for the parameters $\beta = a_0 = 1$ (see the bold curve). The asymptote of this curve is equal to $\sigma_\eta^{\,2} = 4.6708$ as well. We can see that the application of the non optimal Gaussian Algorithm give us acceptable results for the samples which are in the interval $[1 < \eta(T_N) < 2]$. It means that the comparison of the efficiency of both algorithms shows that there are very different reconstruction errors in the considered variants.

6 Conclusions

We demonstrate that the conditional mean rule can be applied for the statistical description of both the optimal and non optimal reconstruction algorithms with respect of non Gaussian processes. The results of this investigation show that we can obtain the principal characteristics of optimal and non optimal SRP process on the output of the converter non lineal of the exponential type. There are some problems for the next investigations: the average operation of the error reconstruction function, the interpolation algorithm of the reconstruction and the evaluation of its efficiency.

References

1. Balakrishnan, A.: A Note on the Sampling Principle for Continuous Signals, IRE Trans, Information Theory, Vol. IT-3. (1957) 143-146.
2. Cramer, H.,: Mathematical Methods of Statistics, Prinston University (1946)
3. Kazakov, V.: The Sampling Reconstruction Procedure with a Limited Number of Samples of Stochastic Processes and Fields on the Basis of the Conditional Mean Rule, Electromagnetic Waves and Electronic System (2005)
4. Kazakov, V., Rodriguez, D.: Study of the Sampling-Reconstruction Procedure of the Gaussian Processes with a Number of Limited Samples: Master Degree Thesis IPN, Mexico (2003)
5. Kazakov, V., Sánchez, S.: Sampling and Reconstruction Procedure of Gaussian Processes with an Arbitrary Number of samples and Uniform Jitter, Electromagnetic Waves and Electronic System, Vol. 9. (2004) 108-114
6. Stratonovich R.L.: Topics in the theory of random noise. Vol.1. Gordon and Breach, N.Y. (1963).
7. Lacaze B.:Periodic Bi-sampling of Stationary Processes. Signal Processing (1998).
8. Marvasti F.:Non Uniform Sampling Theory and Practice Information Technology Processing and Storage (2001) 361-391